On the Navier-Stokes Equations

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$$\rho\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right)=-\nabla p+\mu\nabla^2+f. \qquad (1)$$

Equation (1) represents the governing Navier-Stokes vector equation, which mathematically formulates the conservation of momentum.

In this expression, the rate of change of momentum on the left-hand side is balanced by the net force on the right-hand side, consisting of the pressure gradient, viscous effects, and body forces.

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+f_x. \qquad (2)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+f_y. \qquad (3)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+f_z. \qquad (4)$$

Equations (2), (3), and (4) consolidate the component-wise physical quantities (u, v, w) and differential operators into a vector format using the del operator (∇). This representation encapsulates Newton's laws of motion by expressing the equilibrium between the acceleration in each axial direction and the sum of all applied external forces into a single, concise equation.

$$u\frac{\partial u}{\partial x}=\frac{\partial x}{\partial t}\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t}. \qquad (5)$$

In Equation (5), the convective acceleration term ($u\frac{\partial u}{\partial x}$) represents the material rate of change in velocity experienced by a fluid particle as it translates through the flow field.

As a fundamental component of the momentum conservation equation, this term embodies the physical entity where the spatial velocity gradient—governed by the deterministic laws of fluid mechanics (conservation of energy and momentum)—is manifested as temporal acceleration.

Therefore, the author posits that fluid motion does not occur randomly, but rather proceeds according to the laws of nature and the principle of causality depending on the environment.

Furthermore, since fluid is a continuous medium, a change in momentum at any given point (downstream) instantaneously influences the flow at other points (upstream) through pressure and viscosity.

This interconnectedness ensures that acceleration does not change abruptly or erratically.

Instead, the spatial gradients and temporal changes synchronize into a unified order, as formulated in the proposed simplified equations.

Thus, the apparent complexity of turbulence is, in fact, a manifested chain of causal events where every local movement is fundamentally linked to the global state of the flow field.

$$4\rho\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + f_x. \tag{6}$$

$$4\rho\left(\frac{\partial v}{\partial t}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + f_y. \tag{7}$$

$$4\rho\left(\frac{\partial w}{\partial t}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + f_z. \tag{8}$$

By applying the principles of Equation (5) to Equations (2), (3), and (4), Equations (6), (7), and (8) are derived.

$$\frac{\partial u}{\partial t} = -\frac{1}{4\rho}[\frac{\partial p}{\partial x} - \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - f_x]. \tag{9}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{4\rho}[\frac{\partial p}{\partial y} - \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - f_y]. \tag{10}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{4\rho}[\frac{\partial p}{\partial z} - \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - f_z]. \tag{11}$$

Dividing both sides of Equations (6), (7), and (8) by 4ρ and rearranging the terms yields the final Equations (9), (10), and (11).